Area and Perimeter of soap bubbles

BSSM 2022

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ULB

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Queen Dido and the city of Carthage

The city of Carthage

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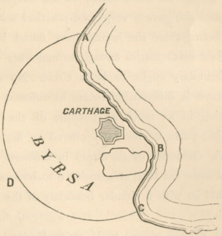

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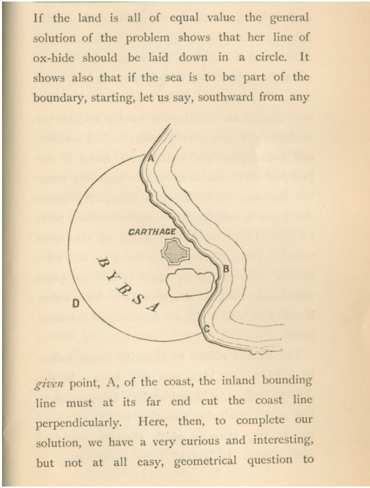
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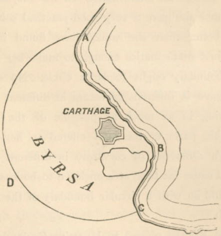

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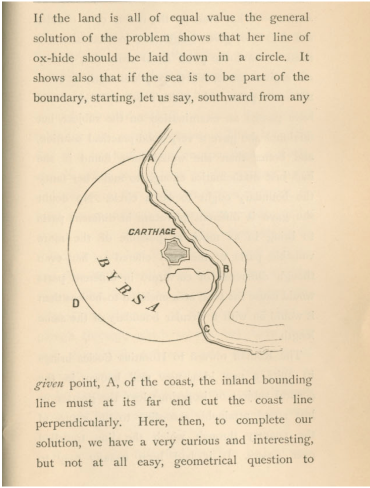
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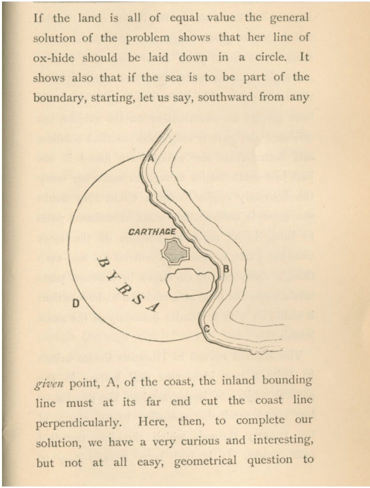
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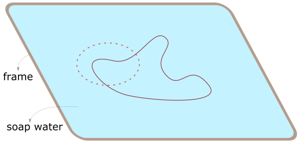
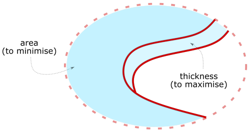
$\longrightarrow$ Isoperimetric problem

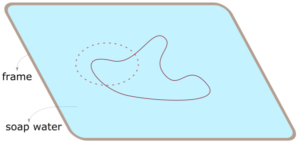
The mathematics of soap films

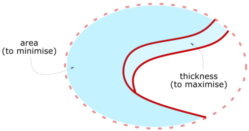




Experiment

The mathematics of soap films





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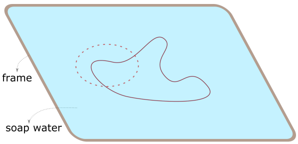

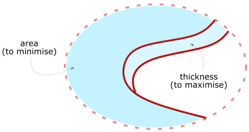

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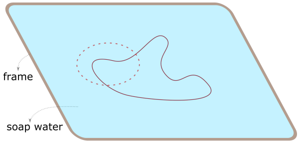

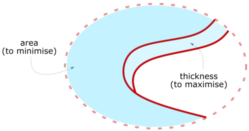

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Moral: Soap films minimise area

Minimal surfaces
= surfaces that *locally* minimise area
= vanishing mean curvature

Isoperimetric inequality

Theorem

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$$A \leq \frac{L^2}{4\pi}.$$

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(cf. talk by Denis Bonneure and François Thilmany in BSSM 2011)

Proof.

Suppose $L = 2\pi$, $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$ by arc-length $\gamma(t) = (x(t), y(t))$:

$$\text{(by Fourier)} \quad x(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}, \quad y(t) = \sum_{n \in \mathbb{Z}} b_n e^{int}$$

$$2\pi = \int_0^{2\pi} |\dot{x}|^2 + |\dot{y}|^2 = \sum n^2(|a_n|^2 + |b_n|^2)$$

$$A = \frac{1}{2}\int_\gamma (xdy - ydx) = \frac{1}{2}\sum in(a_{-n}b_n - a_n b_{-n}) \leq \frac{1}{2}.2\pi = \pi$$

Minimal surfaces

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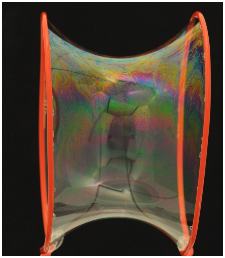
Catenoid



Catenary

Isoperimetric inequality for minimal surfaces

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- Carleman (1921): minimal discs
- Reid (1959), Hsiung (1961): minimal surfaces with connected boundary
- Osserman–Schiffer (1975), Feinberg (1977): minimal annuli
- Li–Schoen–Yau (1984): *weakly connected* boundary
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Today's goal: prove Choe's result.

Strategy

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Σ: minimal surface of $\mathbb{R}^n$, $\gamma = \partial\Sigma$: boundary

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radially connected from O: $\exists O \in \Sigma$ s.t. $\{d(O,x) : x \in \gamma\} \subset \mathbb{R}$ is connected.

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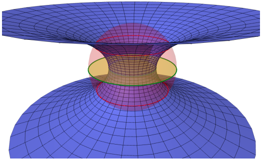
$$A \leq \frac{L^2}{4\pi} \tag{1}$$

2 steps:

- Prove that area of a minimal surface is less than area of the "cone"
- Prove (1) for "cones"

Monotonicity theorem



$\Sigma \subset \mathbb{R}^n$: surface,
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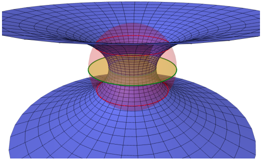

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$Q(r) = \pi r^2$: area of equatorial disc

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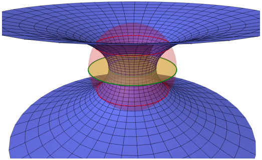

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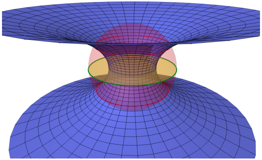

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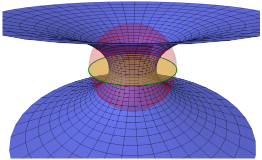

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Corollary

1. *A minimal surface whose boundary is a curve γ has less area than the cone C_γ built uppon γ.*
2. *If $O \in \Sigma$, the point O sees γ with an "angle" $\geq 2\pi$*

Reduce to cones

Choose O to be on the surface. C_γ: cone at O

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Need to prove: $A \leq \frac{L^2}{4\pi}$ for C_γ.
(See blackboard...)

References

- [Choe 1990] *The isoperimetric inequality for a minimal surface with radially connected boundary.*
- [Gromov 1983] *Filling Riemannian manifolds.*
- [Thurston 1997] *Three-dimensional geometry and topology.*
- [White 2013] *Lectures on Minimal Surface Theory.*